THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4060 Complex Analysis
Homework 3 Suggested Solutions
Date: 13 March, 2025

1. (Exercise 7 of Chapter 6 of [SS03]) The **Beta function** is defined for $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$ by
$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt.$$

   (a) Prove that $B(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

   (b) Show that $B(\alpha, \beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$.

   [Hint: For part (a), note that
$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds,$$
   and make the change of variables $s = ur, t = u(1-r)$.]

Solution. (a) Note that with the change of variables given in the hint: $s = ur, t = u(1-r)$, the new bounds are $0 < u < \infty, 0 < r < 1$ and the change of variables matrix is given by
$$\begin{bmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial r} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial r} \end{bmatrix} = \begin{bmatrix} r & u \\ 1-r & -u \end{bmatrix}, \quad \left| \frac{\partial(s,t)}{\partial(u,r)} \right| = u.$$

Applying this transformation to $\Gamma(\alpha)\Gamma(\beta)$, we have
$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds \\ &= \int_0^\infty \int_0^1 (u(1-r))^{\alpha-1} (ur)^{\beta-1} e^{-u(1-r)-ur} u du dr \\ &= \int_0^\infty e^{-u} u^{\alpha+\beta-1} du \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr \\ &= \Gamma(\alpha+\beta) B(\alpha, \beta) \end{aligned}$$

as required.

(b) Note that by applying the change of variables $s = 1-t$ to the defining integral for B, we see that
$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt = \int_0^1 s^{\alpha-1} (1-s)^{\beta-1} ds = B(\beta, \alpha).$$

Consider the change of variables $u = \dfrac{1}{1-s} - 1$. The new bounds of integration are $0 < u < \infty$ and note that

$$u = \frac{1}{1-s} - 1 \Leftrightarrow s = \frac{u}{u+1}, \quad ds = \frac{1}{(u+1)^2}du.$$

Applying this change of variables, we have

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds \\ &= \int_0^\infty \left(\frac{u}{u+1}\right)^{\alpha-1}\left(1-\frac{u}{u+1}\right)^{\beta-1}\frac{1}{(u+1)^2}du \\ &= \int_0^\infty \left(\frac{u}{u+1}\right)^{\alpha-1}\left(\frac{1}{u+1}\right)^{\beta-1}\frac{1}{(u+1)^2}du \\ &= \int_0^\infty \frac{u^{\alpha-1}}{(u+1)^{\alpha+\beta}}du \end{aligned}$$

as required.

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2. (Exercise 11 of Chapter 6 of [SS03]) Let $f(z) = e^{az}e^{-e^z}$ where $a > 0$. Observe that in the strip $\{x+iy : |y| < \pi\}$ the function $f(x+iy)$ is exponentially decreasing as $|x|$ tends to infinity. Prove that

$$\hat{f}(\zeta) = \Gamma(a+i\zeta), \quad \text{for all } \zeta \in \mathbb{R}.$$

Solution. As noted by some students, this question appears to have a few typos: In order for the exponential decay of f to hold, the condition on y should instead be $|y| < \frac{\pi}{2}$. Additionally, the formula should be

$$\hat{f}(\zeta) = \Gamma(a-2\pi i\zeta), \quad \text{for all } \zeta \in \mathbb{R}.$$

We proceed with the solution. Note that for $|y| < \frac{\pi}{2}$, $\cos y < 0$, and so we see that

$$|f(x+iy)| = \left|e^{ax+iay}e^{-e^{x+iy}}\right| = e^{ax}e^{-e^x\cos y} \to 0 \text{ exponentially as } |x| \to +\infty.$$

So we have that $f \in \mathcal{F}_{\pi/2}$. By definition of the Fourier transform, and applying the change of variables $u = e^x$, $u^{-1}du = dx$, we have

$$\begin{aligned} \hat{f}(\zeta) &= \int_{-\infty}^\infty e^{ax}e^{-e^x}e^{-2\pi ix\zeta}dx \\ &= \int_{-\infty}^\infty (e^x)^{a-2\pi i\zeta}e^{-e^x}dx \\ &= \int_0^\infty u^{a-2\pi i\zeta}e^{-u}u^{-1}du \\ &= \int_0^\infty u^{a-2\pi i\zeta-1}e^{-u}du = \Gamma(a-2\pi i\zeta) \end{aligned}$$

as required. ◀

3. (Exercise 13 of Chapter 6 of [SS03]) Prove that

$$\frac{d^2 \log \Gamma(s)}{ds^2} = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}$$

where s is a positive number. Show that if the left-hand side is interpreted as $(\Gamma'/\Gamma)'$, then the above formula also holds for all complex numbers s with $s \neq 0, -1, -2, \ldots$.

Solution. We use the product formula for $\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$ (Theorem 1.7 of Chapter 6 of [SS03]), we have that

$$\begin{aligned} \log \Gamma(s) &= \log \left(e^{-\gamma s} s^{-1} \prod_{n=1}^{\infty} \left(\frac{n}{n+s} \right) e^{s/n} \right) \\ &= -\gamma s - \log s + \sum_{n=1}^{\infty} \left(\frac{s}{n} + \log n - \log (n+s) \right) \end{aligned}$$

where the right-hand side converges for any $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ since $\frac{1}{\Gamma(s)}$ vanishes only at $s \in \{0, -1, -2, \ldots, \}$. Since we have uniform convergence on any compact subset of $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$, we can differentiate term-by-term inside and obtain

$$\begin{aligned} \frac{d}{ds} \log \Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)} &= \frac{d}{ds} \left(-\gamma s - \log s + \sum_{n=1}^{\infty} \left(\frac{s}{n} + \log n - \log (n+s) \right) \right) \\ &= -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+s} \right). \end{aligned}$$

Since the sum $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+s}\right)$ is telescoping, we can show that it converges uniformly and again we differentiate term-by-term to obtain

$$\begin{aligned} \frac{d^2}{ds^2} \log \Gamma(s) &= \frac{d}{ds} \left(-\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+s} \right) \right) \\ &= \frac{1}{s^2} + \sum_{n=1}^{\infty} \frac{1}{(n+s)^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+s)^2} \end{aligned}$$

as required. ◀

4. (Exercise 15 of Chapter 6 of [SS03]) Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$

[Hint: Write $1/(e^x - 1) = \sum_{n=0}^{\infty} e^{-nx}$.]

Solution. Following the hint, we have

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \int_0^\infty \sum_{n=0}^\infty e^{-nx} x^{s-1} dx$$

then since for $\text{Re}(s) > 1$ the integral and sum are absolutely integrable and absolutely convergent, we can apply Fubini's theorem to interchange the order of the sum and integral and obtain

$$\begin{aligned} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx &= \int_0^\infty \sum_{n=0}^\infty e^{-nx} x^{s-1} dx \\ &= \sum_{n=0}^\infty \int_0^\infty e^{-nx} x^s x^{-1} dx \\ &= \sum_{n=0}^\infty \int_0^\infty e^{-y} y^s n^{-s} y^{-1} n n^{-1} dy \\ &= \zeta(s) \int_0^\infty e^{-y} y^{s-1} dy = \zeta(s)\Gamma(s) \end{aligned}$$

where we used the change of variables $y = nx$. ◀

References

[SS03] Elias M. Stein and Rami Shakarchi. *Complex Analysis*. Princeton University Press, 2003.